Arbitrary Size Benes Networks

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Abstract

The Benes network is a rearrangeable nonblocking network which can realize any arbitrary permutation. Overall, the r-dimensional Benes network connects 2^r inputs to 2^r outputs through $2r-1$ levels of 2×2 switches. Each level of switches consists of 2^{r-1} switches, and hence the size of the network has to be a power of two. In this paper, we extend Benes networks to arbitrary sizes. We also show that the looping routing algorithm used in Benes networks can be slightly modified and applied to arbitrary size Benes networks.

1 Introduction

A multistage network consists of more than one stage of switching elements and is usually capable of connecting an arbitrary input terminal to an arbitrary output terminal. Multistage networks are classified into blocking, rearrangeable, or nonblocking networks. In blocking networks, simultaneous connections of more than one terminal pair may result in conflicts in the use of network communication links. A network is a rearrangeable nonblocking network if it can realize all possible permutations between inputs and outputs. However, if the connections in a permutation are established in the network sequentially, the establishment of a connection may require rearranging the existing connections. A network which can handle all possible permutations without rearranging connections is a nonblocking network [3].

The Benes network [1], which is a special instance of CLOS networks [2], is an excellent example of a rearrangeable network. Overall, the r-dimensional Benes network has $2r-1$ levels of switches, with 2^{r-1} switches in each level. Figure 1 shows the Benes network with $r = 3$. Given any one-to-one mapping, Π, of 2^r inputs to 2^r outputs, there is a set of edge-disjoint paths from the inputs of an r-dimensional Benes network to its outputs connecting input i to output $\Pi(i)$ for $0 \leq i \leq 2^r - 1$ [1, 4].

The Benes topology is specified such that the number of input or output terminals has to be a power of 2. In practical terms, this is a severe restriction on the sizes of systems that will use the network. If the size of the needed network is not a power of 2, a larger than needed network has to be used, and many of the resources in the used network will remain idle. In this paper, we present a constructive way of building an arbitrary size Benes network (AS-Benes) for any number of terminals. The routing algorithm presented for the AS-Benes is nearly as simple as the looping algorithm for the regular Benes network [5].

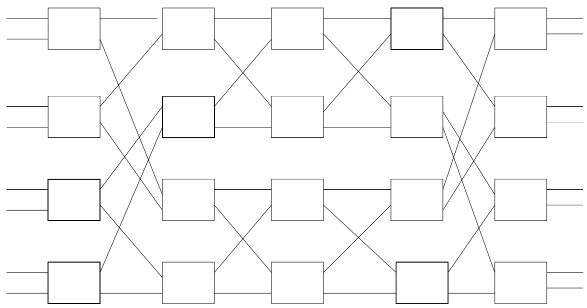


Figure 1: A 8 x 8 Benes network with $r = 3$

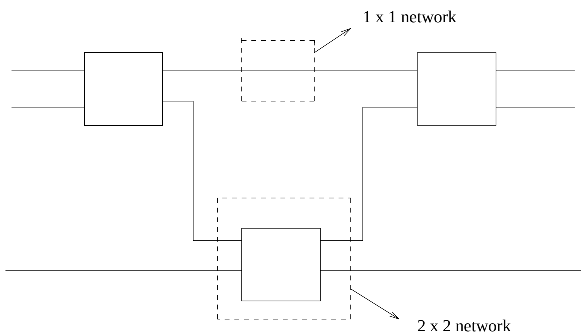


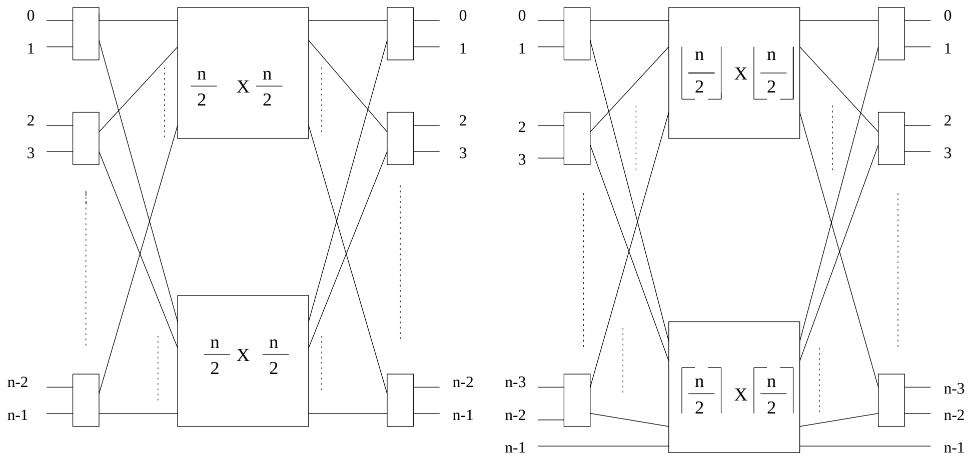
Figure 2: A 3 x 3 AS-Benes network

2 Construction Strategy

Multistage interconnection networks are usually constructed from a single type of modular switching elements. Each element is a 2×2 switch which can be set by a control line into a direct-connection state or a crossed-connection state, thus realizing all permutations from two inputs to two outputs. Three 2×2 switches can be used to construct a network which can relize any 3×3 permutation as shown in Figure 2. If we consider a simple wire to be a network that can realize any 1×1 permutation, we can view the 3×3 network in Figure 2 as being built from a 2×2 network and a 1×1 network.

The procedure used to construct a network of size 3 can be generalized to recursively construct a network of any size. Specifically, an AS-Benes of size n is constructed recursively from an AS-Benes of size $\lfloor \frac{n}{2} \rfloor$ and an AS-Benes of size $\lceil \frac{n}{2} \rceil$. When n is even, the construction is similar to that of the Benes network where the n inputs are connected to $\frac{n}{2}$ switches and each switch is connected to two AS-Benes networks of size $\frac{n}{2}$. Similarly, the n outputs are connected to $\frac{n}{2}$ switches and each switch is connected to the two $\frac{n}{2}$ AS-Benes networks (see Figure 3(a)).

To construct an AS-Benes of an odd size, the first $n-1$ inputs are connected to $\lfloor \frac{n}{2} \rfloor$ switches and each switch is connected to the AS-Benes of size $\lfloor \frac{n}{2} \rfloor$ and the AS-Benes of size $\lceil \frac{n}{2} \rceil$. Similarly, the first $n-1$ outputs are connected to $\lfloor \frac{n}{2} \rfloor$ switches and each switch is connected



(a) n even

(b) n odd

Figure 3: Constructions of AS-Benes networks

to the two AS-Benes networks. The last input and the last output are connected directly to the $\lceil \frac{n}{2} \rceil$ AS-Benes as shown in Figure 3(b). This process is illustrated in Figure 4 where an AS-Benes of size 5 is built from an AS-Benes of size 2 and an AS-Benes of size 3. To build an AS-Benes of size 6, two size 3 AS-Benes can be used, and in general, an AS-Benes of size n, for any n, may be constructed.

3 Routing Algorithm

A quick inspection of Figure 3 reveals that, except for paths involving the last input and/or the last output of odd size networks, a path between an input and an output may be established through either the upper AS-Benes sub-network (of size $\lfloor \frac{n}{2} \rfloor$) or the lower AS-Benes sub-network (of size $\lceil \frac{n}{2} \rceil$). Given that each switch at the first and last levels in an AS-Benes has precisely one connection to each of the upper and lower sub-networks, the realization of any given permutation, Π, in an AS-Benes should satisfy the property that paths sharing any switch at the first or last levels must go to different sub-networks. By enforcing this property, it can be shown that, given any one-to-one mapping, Π, of n inputs to n outputs, there is a set of edge-disjoint paths from the inputs of a size n AS-Benes to its outputs connecting input i to output $\Pi(\mathrm{i})$ for $0 \leq i \leq n-1$.

As an example, we illustrate the paths in Figure 5 for the mapping

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 8 & 6 & 2 & 1 & 0 & 3 & 5 \end{pmatrix}$$

in a 9×9 AS-Benes network. The bold paths represent the first loop which starts at input $n-1$ and terminates at output $n-1$. After this loop, there are only two pairs of input/output

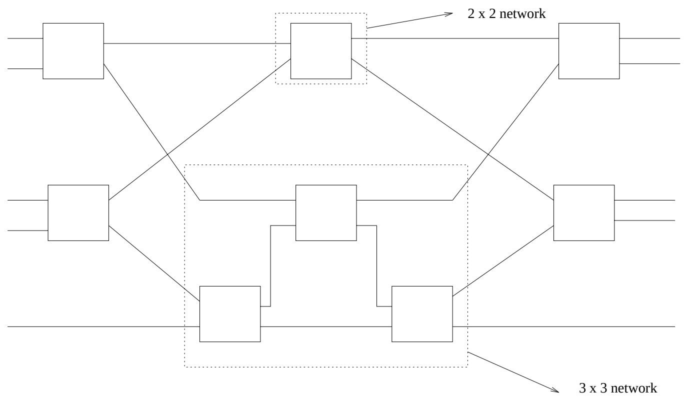


Figure 4: A 5 x 5 Benes network

left which form a second loop. In this way, all paths can be assigned to the upper or lower sub-networks without conflict.

4 Comparison with Benes Networks

It is important to be able to construct rearrangeable networks of arbitrary sizes with minimum cost. In this section, we compare the number of switches used for an AS-Benes of size n with a Benes of size $2^{\lceil \log n \rceil}$, which is the smallest Benes that can realize any $n \times n$ permutation. For instance, to realize any 5×5 permutations, an 8×8 Benes is needed, which requires twenty 2×2 switches. A 5×5 AS-Benes requires only eight 2×2 switches (see Figure 4). That is more than 100% saving. In general, if $S(k)$ is the number of switches used for a size k AS-Benes, then $S(1) = 0$, $S(2) = 1$ and

$$S(k) \quad = \quad 2\lfloor \frac{k}{2} \rfloor + S(\lceil \frac{k}{2} \rceil) + S(\lfloor \frac{k}{2} \rfloor).$$

It is easy to use induction to prove that the solution to the above equation satisfies $S(k) \leq \frac{k}{2}(2\ logk - 1)$. That is the number of switches in AS-Benes is of order $O(k\ logk)$. The recursive equation for $S(k)$ may be also used to compare the number of switches needed in an $n \times n$ AS-Benes, and an $2^{\lceil \log n \rceil} \times 2^{\lceil \log n \rceil}$ Benes. This comparison is shown in Figure 6 for n up to 32. Clearly, when n is a power of 2, AS-Benes is identical to a Benes. The curves follow similar trends for larger values of n.

Finally, we want to point out that different paths in an AS-Benes may pass through different number of switches. However, the maximum length of any path will never exceed the length of a path in a Benes of size $2^{\lceil \log n \rceil}$. In other word, the delay in an AS-Benes is at most equal to the delay in the corresponding Benes network.

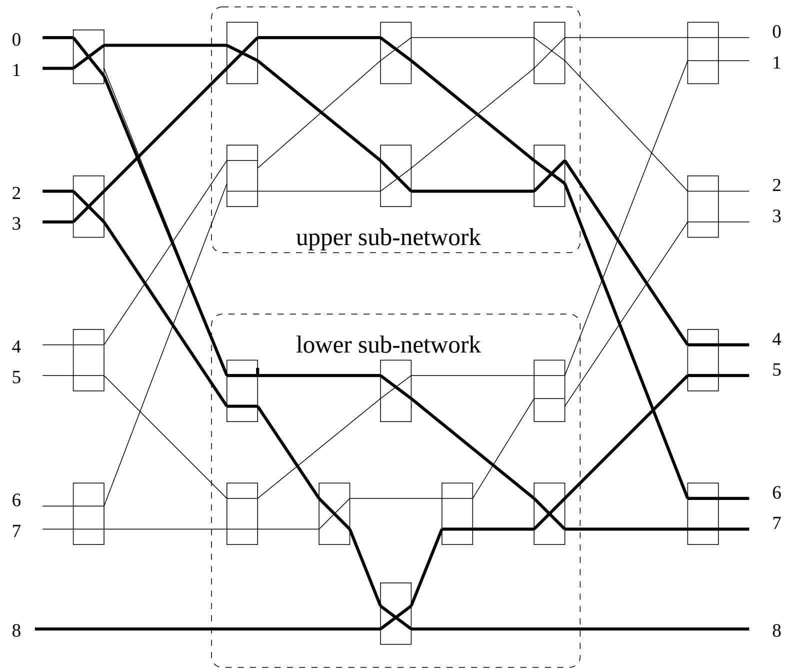


Figure 5: Two loops in the realization of a permutation in a 9×9 AS-Benes

5 Conclusions

We have shown that there is a simple and efficient way for building arbitrary size re-arrangeable networks of size n using $O(n\ logn)$ two by two switches, and for routing permutations in such networks.

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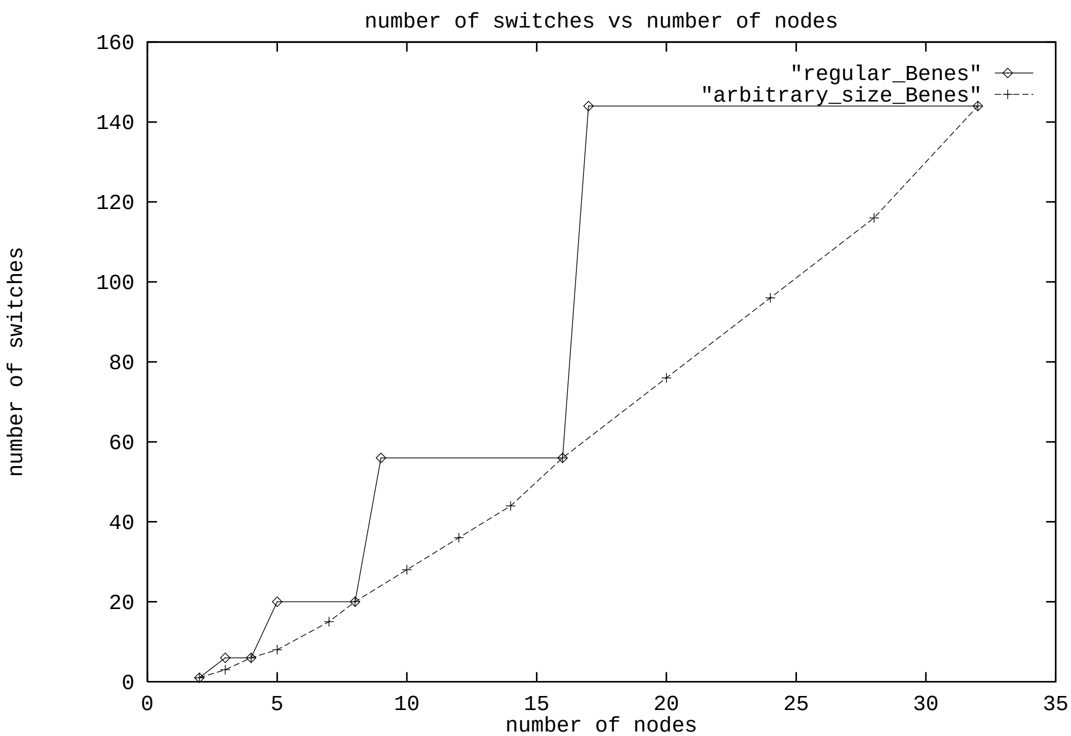

Figure 6: Comparisons between AS-Benes and Benes networks